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LOCAL SOLVABILITY OF SECOND ORDER FUCHSIAN TYPE EQUATIONS (Collaboration with C.PARENTI)

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§1. DISCUSSION IN A EXAMPLE.

Let $(t,x)\in \mathbf{R}_t\times\mathbf{R}_x^n$ and let us consider

$$P=t\partial_t^2-\Delta_x+a(t,x)\partial_t+\Sigma_{i=1}^n b_i(t,x)\partial_i+c(t,x) \qquad (1.1)$$

near the origin, where Δ_x is the Laplacian, $\partial_t=\partial/\partial t$ and $\partial_i=\partial/\partial x_i$ $(1\leq i\leq n)$. Note that P is hyperbolic on $\{t>0\}$ and elliptic on $\{t<0\}$, that is, P is so-called a mixed type operator.

To illustrate our argument, let us treat here the above operator P and show the local solvability of Pu=f in hyperfunctions & distributions.

1-1. Solvability in hyperfunctions. Let P be the operator in (1.1) and assume that $a(t,x)$, $b_i(t,x)$ and $c(t,x)$ are analytic functions near the origin. Let $\mathcal{B}$ be the stalk of sheaf of hyperfunctions at the origin. Then, we have:

Theorem 1. If $1-a(0,x)\notin\{1,2,\ldots\}$, the map $P:\mathcal{B}\longrightarrow\mathcal{B}$ is surjective.

Sketch of proof. To prove this, it is sufficient to show the following:

(i) P : $\mathcal{A} \longrightarrow \mathcal{A}$ is surjective,

(ii) P : $\mathcal{B}/\mathcal{A} \longrightarrow \mathcal{B}/\mathcal{A}$ is surjective.

(i) is clear from the Cauchy-Kowalewski type theorem for Fuchsian type equations. (ii) follows from the following two facts:

(ii-1) when $p(\pm)=(0,0;\tau=\pm 1,\xi=0)$, P : $\mathcal{C}_{p(\pm)} \longrightarrow \mathcal{C}_{p(\pm)}$ is surjective,

(ii-2) when $q=(0,0;\tau,\xi\neq 0)$, P : $\mathcal{C}_q \longrightarrow \mathcal{C}_q$ is bijective,

where $\mathcal{C}_p$ denotes the stalk of sheaf of microfunctions at p.

Q.E.D.

<u>1-2</u>. Solvability in distributions. Let P be the operator in (1.1) and assume that $a(t,x)$, $b_i(t,x)$ and $c(t,x)$ are C^∞ functions near the origin. Let $\mathcal{D}'$ be the stalk of sheaf of distributions at the origin. Then, we have:

<u>Theorem 2</u>. If $1-a(0,x)\notin \mathbf{Z}$, the map P : $\mathcal{D}' \longrightarrow \mathcal{D}'$ is surjective.

<u>Sketch of proof</u>. Put

$$\mathcal{D}'|_{t>0} = \{ u|_{t>0} \ ; \ u\in \mathcal{D}' \},$$
$$\mathcal{D}'|_{t<0} = \{ u|_{t<0} \ ; \ u\in \mathcal{D}' \}.$$

Then, Theorem 2 follows from the following three facts:

(i) P : $\mathcal{D}'|_{t>0} \longrightarrow \mathcal{D}'|_{t>0}$ is surjective,

(ii) P : $\mathcal{D}'|_{t<0} \longrightarrow \mathcal{D}'|_{t<0}$ is surjective,

(iii) for any $\varphi(x)\in\mathcal{D}'$, there exists a unique solution $u(t,x)\in C^\infty([0,T],\mathcal{D}')$ such that

$$Pu=0 \quad \text{on } t>0, \quad u|_{t=0}=\varphi(x).$$

In fact, we can get Theorem 2 as follows. Let $f\in\mathcal{D}'$. Then, by (i) and (ii) we can choose $v\in\mathcal{D}'$ such that $Pv=f+\delta(t)\otimes\psi(x)$. By (iii) we solve

$$\begin{cases} Pw=0 \quad \text{on } t>0, \\ w|_{t=0}=(a(0,x)-1)^{-1}\psi(x). \end{cases}$$

Then, we have $P(Y(t)w)=\delta(t)\otimes\psi(x)$. Thus, by putting $u=v-Y(t)w$ we get a solution $u\in\mathcal{D}'$ of $Pu=f$. Q.E.D.

Remark. By the same argument as in the proof of Theorem 1, we can easily get that $P:\mathcal{D}'/C^\infty \longrightarrow \mathcal{D}'/C^\infty$ is surjective. But, unfortunately, we do not know whether $P:C^\infty \longrightarrow C^\infty$ is surjective or not. This is the reason why we proved Theorem 2 in a different way from the proof of Theorem 1.

§2. FURTHER RESULTS.

Let us treat here somewhat more general Fuchsian type operators. Let P_1, P_2 and P_3 be of the form

$$P_1 = t\partial_t^2-t^kA(t,x,\partial_x)+a(t,x)\partial_t +t^h\Sigma_{i=1}^n b_i(t,x)\partial_i+c(t,x),$$

$$P_2 = t^2\partial_t^2-t^pA(t,x,\partial_x)+a(t,x)t\partial_t +t^q\Sigma_{i=1}^n b_i(t,x)\partial_i+c(t,x),$$

$$P_3 = t^2\partial_t^2+t^pA(t,x,\partial_x)+a(t,x)t\partial_t +t^q\Sigma_{i=1}^n b_i(t,x)\partial_i+c(t,x),$$

where $\partial_t=\partial/\partial t$, $\partial_i=\partial/\partial x_i$ $(1\le i\le n)$, $k,h,p,q\in\mathbf{Z}_+(=\{0,1,2,\dots\})$,

$$A(t,x,\partial_x) = \Sigma_{i,j=1}^n a_{ij}(t,x)\partial_i\partial_j$$

is a real elliptic differential operator such that

$$A(t,x,\xi/|\xi|) > 0 \quad \text{for } \forall(t,x),\ \forall\xi\in R^n\setminus\{0\},$$

and $a_{ij}(t,x)$, $a(t,x)$, $b_i(t,x)$ and $c(t,x)$ are C^∞ functions near the origin. Let $\rho(x)=1-a(0,x)$ and let $\rho_1(x),\rho_2(x)$ be the roots of $\rho(\rho-1)+a(0,x)\rho+c(0,x)=0$. Note that $\rho(x)$ is the non-trivial characteristic exponent of P_1 and that $\rho_1(x),\rho_2(x)$ are the characteristic exponents of P_2 and P_3. Then, we have:

<u>Theorem 3</u>. (1) In case P_1, if $k\geq 0$, $h\geq(k-1)/2$ and $\rho(0)\notin \mathbf{Z}$ hold, the map $P_1 : \mathcal{D}'\longrightarrow\mathcal{D}'$ is surjective.

(2) In case P_2, if $p\geq 1$, $q\geq p/2$ and $\rho_1(0),\rho_2(0)\notin\{-1,-2,\dots\}$, the map $P_2 : \mathcal{D}'\longrightarrow\mathcal{D}'$ is surjective.

(3) In case P_3, if $p\geq 1$, $q\geq p/2$ and $\rho_1(0),\rho_2(0)\notin\{-1,-2,\dots\}$, the map $P_3 : \mathcal{D}'\longrightarrow\mathcal{D}'$ is surjective.

The proof of this result is quite similar to that of Theorem 2.

REFERENCES

[1] A.Bove, J.E.Lewis, C.Parenti and H.Tahara : Cauchy problem for Fuchsian hyperbolic operators, II. J. Fac. Sci. Univ. Tokyo, 34-1 (1987).

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要約

本稿では、2階のフックス型線形偏微分方程式の hyper-function ($\mathcal{B}$) 或いは distribution ($\mathcal{D}'$) での局所可解性が論じられている。

① $(t,x)\in \mathbb{R}_t\times\mathbb{R}^n_x$（原点の近傍）とし、例えば

$$P = t\partial_t^2 - \Delta_x + a(t,x)\partial_t + \sum_{i=1}^n b_i(t,x)\partial_{x_i} + c(t,x)$$

という作用素を考えてみる。　P は $\{t>0\}$ では双曲型であり $\{t<0\}$ では楕円型となっていて、いわゆる混合型作用素と呼ばれているものになる。　次が成り立つ。

(i) P の係数が解析的とする。この時、原点の近傍で

$1-a(0,x)\notin\{1,2,\cdots\}$ $\Rightarrow$ $P:\mathcal{B}\to\mathcal{B}$ は全射。

(ii) P の係数が C^∞ クラスとする。この時、原点の近傍で

$1-a(0,x)\notin\{0,\pm1,\pm2,\cdots\}$ $\Rightarrow$ $P:\mathcal{D}'\to\mathcal{D}'$ は全射。

② (ii) の証明は L^2-評価をベースとした議論による。同様の議論によって　次の様な作用素の $\mathcal{D}'$ での局所可解性を扱かうことも出来る。

$$\begin{aligned} P_1 &= t\partial_t^2 - t^k A(t,x,\partial_x) \\ &\qquad + a(t,x)\partial_t + t^h \sum_{i=1}^n b_i(t,x)\partial_{x_i} + c(t,x), \\ P_2 &= t^2\partial_t^2 - t^p A(t,x,\partial_x) \\ &\qquad + a(t,x)t\partial_t + t^q \sum_{i=1}^n b_i(t,x)\partial_{x_i} + c(t,x), \\ P_3 &= t^2\partial_t^2 + t^p A(t,x,\partial_x) \\ &\qquad + a(t,x)t\partial_t + t^q \sum_{i=1}^n b_i(t,x)\partial_{x_i} + c(t,x). \end{aligned}$$

但し、$A(t,x,\partial_x)$ は 2階の実係数楕円型作用素、$k,h,p,q\in \mathbb{Z}_+(=\{0,1,2,\cdots\})$ で　$h\geq(k-1)/2$, $p\geq 1$, $q\geq p/2$　とする。